The Fundamental Financial Equation.

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Abstract:
The Fundamental Financial Equation link the 5 variables. Present value, Future value, period Payment, number of period and the interest rate together in this equation:

\[
(PV + \frac{PMT(1 + cIR)}{IR})((1 + IR)^{NP} - 1) + PV + FV = 0
\]

That can be useful in a number of scenario including mortgage calculation. This paper highlight the equation and devise a method for how to calculate all the other variables when only 4 of the 5 variables are known, including the Interest rate based on Newton iteration.
The Fundamental Financial Equation

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Introduction to the Fundamental Financial Equation

The Fundamental Financial equation is a financial equation that links the Present Value (PV) of a loan, investment, starting balance etc., with the Future Value (FV), with a periodical payment (PMT), with the number of periods (NP) to an interest rate of (IR) through the equation:

$$
PV + \frac{PMT(1+cIR)}{IR}((1 + IR)^{NP} - 1) + PV + FV = 0
$$

(1)

The variables are:

- PV – Present Value of a starting balance of a bank account, Mortgage, Investment etc.
- FV – Future Value of above starting balance
- NP – Number of periods. Can be Years, quarters, months or daily.
- PMT – Periodic payment
- IR – Interest rate per period.
- c – Is a constant that is 1 for interest add at the beginning of a period and 0 for the end of the period.

The PV, FV, PMT can either be 0, positive or negative. The sign is used to either indicating taking it out of the account (positive), negative for paying into the account. The result are e the same regardless as long as it is done consistently. So reversing the sign of PV, FV and PMT yield the same result.

Above formula can also be written as:

$$
PV(1 + IR)^{NP} + PMT(1 + cIR)((1 + IR)^{NP} - 1)/IR + FV = 0
$$

(2)

Or

$$
\frac{PV \times IR(1 + IR)^{NP} + PMT(1 + cIR)((1 + IR)^{NP} - 1) + FV \times IR}{IR} = 0
$$

And other variations. But the basic underline equation are all the same.

Now by knowing 4 out of the 5 variables you can always calculate the 5th variable.

In order to make it easy we create the following entity:

$$
A = (1 + IR)^{NP} - 1
$$

(3)
The Fundamental Financial Equation

And

\[ B = \frac{1 + c \cdot IR}{IR} = \frac{1}{IR} + c \]  \hspace{1cm} (4)

Substituting (3) and (4) above into (1) you get:

\[ PV + (PMT \cdot B) \cdot A + PV + FV = 0 \]  \hspace{1cm} (5)

Which solves for each variable gives the following equations:

\[ PV = -\frac{FV + A \cdot PMT \cdot B}{A + 1} \hspace{1cm} \text{Conditions IR} \neq -1 (-100\%) \]  \hspace{1cm} (6)

\[ FV = -(PV + A(PV + PMT \cdot B)) \]  \hspace{1cm} (7)

\[ PMT = -\frac{FV + PV(A + 1)}{A \cdot B} \hspace{1cm} \text{Condition} \hspace{0.5cm} IR \neq -1 (-100\%) \hspace{0.5cm} \text{an} \hspace{0.5cm} c = 1 \]  \hspace{1cm} (8)

\[ NP = \frac{\log \left( \frac{PMT \cdot B - FV}{PMT \cdot B + PV} \right)}{\log(1 + IR)} = \frac{\log(PMT \cdot B - FV) - \log(PMT \cdot B + PV)}{\log(1 + IR)} \]  \hspace{1cm} (9)

And then unfortunately we would need to resort to iteration to find the IR based on our fundamental financial equation. More about that later on.

Looking at (1) we see that the only troubling part is the division with IR when IR=0. we will take a closer look at that in special case 1 below.

Looking at (6) we see that we divide with A+1 however \( A + 1 = (1 + IR)^{NP} - 1 + 1 = (1 + IR)^{NP} \) which is always <> 0 expect for IR=-1. In (8) we have a similar problem since we divide by A*B meaning that neither A or B can be zero. A is zero for IR=0 since \( A = (1 + IR)^{NP} - 1 = (1 + 0)^{NP} - 1 = 0 \). B can be zero only for IR=-1 and c=1. For IR=0 it is undefined but that was already stated that (3) only works for IR<>0.

Lastly in (9) we have that neither \( \log(1+IR)<0 \) or \( \log \left( \frac{PMT \cdot B - FV}{PMT \cdot B + PV} \right) < 0 \). In the first case we have that IR=0 yields \( \log(1+0)=0 \) and for IR<=-1 \( \log \) is undefined since that the argument to the \( \log() \) is <=0 where the \( \log() \) function is undefined at least in the real domain. In the second case we need \( \frac{PMT \cdot B - FV}{PMT \cdot B + PV} > 0 \) otherwise NP can’t be calculated.
**The Fundamental Financial Equation**

**Special case 1 (IR=0, PMT <> 0):**

Now let’s consider a special case where IR=0 and PMT is not 0 and we quite easily find

\[ PV + NP \times PMT + FV = 0 \]  \hspace{1cm} (10)

Which solves for PV, FV, PMT and NP gives:

\[ PV = -(FV + NP \times PMT) \]  \hspace{1cm} (11)

\[ FV = -(PV + NR \times PMT) \]  \hspace{1cm} (12)

\[ PMT = - \frac{PV+FV}{NP} \]  \hspace{1cm} (13)

\[ NP = - \frac{PV+FV}{PMT} \]  \hspace{1cm} (14)

**Special case 2 (IR<>0, PMT=0)**

Another special case is where IR is not equal to 0 and PMT is 0:

\[(PV(1+IR)^{NP} - 1) + PV + FV = 0\]  \hspace{1cm} (15)

Which solves for PV, FV, NP and IR gives using the substitution at (3):

\[ PV = - \frac{FV}{1+A} \]  \hspace{1cm} (16)

\[ FV = -PV(1+A) \]  \hspace{1cm} (17)

\[ NP = \frac{\log\left(\frac{FV}{PV}\right)}{\log(1+IR)} \]  \hspace{1cm} (18)

\[ IR = \left(\frac{FV}{PV}\right)^{\frac{1}{NP}} - 1; \text{ Condition: } PV \times FV < 0 \]  \hspace{1cm} (19)

**Special case 3 (when IR=0 and PMT=0)**

This case is not really relevant since you basically have PV+FV=0 and regardless of number of period you will also have that PV=-FV or FV=-PV
Interest Compounding & Period Payment Frequency

The interest rate IR is usually specified in annual interest rate in %. However as you have seen from your mortgage these are not really compounding on an annually basic, but rather on a monthly, quarterly or even on even continuous compounding rate. We have also learn from the period payment PMT that typically for your mortgage is done on a monthly basic, but other payment rate can also been seen as quarterly, half year etc. so we will introduce two extra variable Periodic Payment frequency PF and Interest Compounding frequency CF and then distinguish between continuous compounding and discrete compounding with the frequency of CF.

In order to convert from the nominal interest (IN) or annually interest rate to the IR in our formula (sometimes it is called the effective interest rate) we use the conversion below:

Continuous compounding:

\[
IR = \exp \left( \frac{IN}{PF} \right) - 1
\]  

Where IN is the nominal or annually rate of interest.

For discrete compounding we need to take both the Compounding frequency CF and Payment frequency PF into account:

\[
IR = \left( 1 + \frac{IN}{CF} \right)^{\frac{PF}{CF}} - 1
\]  

And to convert back to the nominal interest IN we get:
Continuous compounding:

\[
IN = \ln((1 + IR)^{PF})
\]  

And for discrete compounding:

\[
IN = CF((1 + IR)^{\frac{PF}{CF}} - 1)
\]

Example for IR calculation

Let’s see how it works out using an example. Assuming you use discrete compounding and a frequency for both CF and PF as monthly and a nominal interest rate IN=6%. Since there is 12 months in a year you have:

\[
PF = 12, CF = 12, IN = 6\% = 0.06
\]
The Fundamental Financial Equation

Using (21) you get:

$$IR = \left(1 + \frac{0.04}{12}\right)^{12} - 1 = 0.005$$

And to convert it back using

$$IR = 0.005 = 12, CF = 12, PF = 12$$

And discrete compounding (23) you get:

$$IN = 12 \left(1 + \frac{0.005}{12}\right)^{12} - 1 = 0.06 = 6\%$$

Another example for using continuous compounding using:

$$PF = 12, IN = 8\% = 0.08$$

You get:

$$IR = \exp\left(\frac{0.08}{12}\right) - 1 = 0.0067$$

And to convert it back using (22) you get:

$$IN = \ln((1 + 0.0067)^{12}) = 0.08 = 8\%$$

**Example of Mortgage calculation of the monthly payment**

What is your monthly payment on a fixed 30 years mortgage of $200,000 at an annually nominal rate of 6%? Since it is compounding monthly and interest rate is accrued monthly at the end we have

$$PF = 12, CF = 12, IN = 6\%, c = 0, PV = 200,000, FV = 0, NP = 30 * 12 = 360$$

We have in the previous example already calculated the IR=0.005

Using formula (8): 

$$PMT = -\frac{FV + PV(A+1)}{A+B}$$

and A is given by (3) and B in (4) we get:

$$A = (1 + 0.005)^{360} - 1 = 5.0225$$

$$B = \frac{1+0+0.005}{0.005} = 200$$

$$PMT = -\frac{0+200\times000(5.0225+1)}{5.0225\times200} = 1199.10$$

The monthly payment PMT is $1,199.10.
If we change it to 20 years mortgage instead of 30 years we get:

\[ NP = 20 \times 12 = 240 \]

\[ A = (1 + 0.005)^{240} - 1 = 2.3102 \]

\[ B = \frac{1 + 0.005}{0.005} = 200 \]

\[ PMT = -0.02 \frac{2.3102 + 200}{2.3102 + 200} = -1432.86 \]

The new monthly payment is $1432.86

**Example of Mortgage calculation with a balloon payment at the end**

Same example as the last one however instead of paying a monthly payment of $1,432.86 over 20 years you have negotiated a lower monthly rate of $1,300 and instead have to pay a balloon payment at the end after 20 years. What is the value of the balloon payment at the end?

\[ PF = 12, CF = 12, IN = 6\%, c = 0, PV = 200,000, PMT = -1300, NP = 20 \times 12 = 240 \]

FV is giving by (7) as:

\[ FV = -(PV + A(PV + PMT \times B)) \]

\[ A \& B \text{ has previous been found to be } A=2.3102 \text{ and } B=200 \text{ so we have:} \]

\[ FV = -(200,000 + 2.3102(200,000 - 1300 \times 200)) = -61,388 \]

The minus sign indicate that you owe the mortgage company $61,388 after 20 years. (The correct result is $61,387.73 but we only used 4 digits in the calculation of A).

**Example of Mortgage calculation with number of period**

What is your monthly payment on a fixed 30 years mortgage of $200,000 at an annually nominal rate of 6%? Since it is compounding monthly and interest rate is accrued monthly at the end we have

\[ PF = 12, CF = 12, IN = 6\%, c = 0, PV = 200,000, FV = 0, NP = 30 \times 12 = 360 \]

We have in a previous example calculated the monthly payment to be $1199.10

Now what happen if we instead increase our monthly payment to $1300?
We have in the previous example already calculated the IR=0.005 and the new PMT is -1300.

Using formula (9): \[ NP = \frac{\log\left(\frac{PMT-B-FV}{PMT-B+FV}\right)}{\log(1+IR)} \] and B is given in (4) we get:

\[ B = \frac{1+0.005}{0.005} = 200 \]

\[ NP = \frac{\log\left(\frac{-1300 \times 200 - 0}{-1300 \times 200 + 200000}\right)}{\log(1 + 0.005)} = \frac{\log(4.3333)}{\log(1.005)} = 294 \]

The new number of period is now 294 months or a shortening of the 30 years fixed mortgage with 5½ years.

**Finding the nominal interest rate per year through iteration**

This is the most interesting chapter and the reason I wrote this paper. In our fundamental equation:

\[ PV + \frac{PMT(1 + cIR)}{IR} \left((1 + IR)^{NP} - 1\right) + PV + FV = 0 \]

There is no way to find the interest IR as an equation and instead to solve the equation for IR we need to resort to the classic Newton iterations. Newton Iterations is defined as a number of repeated used of below iterations.

\[ x_{n+1} = x_n - \frac{f(x)}{f'(x)} \text{ Or } x_{n+1} = x_n - d_n \text{ where } d_n = \frac{f(x)}{f'(x)} \]

Now before we can use this we first need to rework our equation so it will be easier to apply the Newton iteration. Substitute x for IR and name IR=i we get:

\[ f(i) = \left(PV + \frac{PMT(1 + ci)}{i}\right) \left((1 + i)^{NP} - 1\right) + PV + FV \]

Which can also be rewritten into an alternative form as:

\[ f(i) = \frac{PMT(1 + ci)(1 + i)^{NP} - PMT(1 + ci) + iFV + iP(1 + i)^{NP}}{i} \]

Now with f(i) above we can find the derivate as:
The Fundamental Financial Equation

\[ f'(i) = \frac{((1 + i)^{NP} - 1)(-PMT(1 + ci)) + ni(1 + i)^{(NP-1)} (PMT(1 + ci) + iPV)}{i^2} \]

We therefore get:

\[ i_{n+1} = i_n + \frac{PMT(1+ci)((1+i)^{NP} - PMT(1+ci)+iFV+iPV(1+i)^{NP}}{i_n((1+i)^{NP-1})(-PMT(1+ci))+ni(1+i)^{(NP-1)}(PMT(1+ci)+iPV)} \]

Or

\[ i_{n+1} = i_n + \frac{PMT(1+ci)((1+i)^{NP} - PMT(1+ci)+iFV+iPV(1+i)^{NP}}{i_n((1+i)^{NP-1})(-PMT(1+ci))+ni(1+i)^{(NP-1)}(PMT(1+ci)+iPV)} \]

To see how it works in real life we can try to find interest from the following scenario.

Example: A 30 year maturity bond with a face value of $1000. Coupon rate of 8% annually. It sells for $900. Interest is accrued at the end of the period. We therefore have

\[ NP = 30, FV = 1000, PV = -900, PMT = 8\% \text{ of } $1000 = 80, \]

Starting with guess of \( i_0 = 1\% \) and get the following iterations.

<table>
<thead>
<tr>
<th>n</th>
<th>i</th>
<th>F(i)</th>
<th>F'(i)</th>
<th>dx</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000%</td>
<td>2,569.73</td>
<td>5,970</td>
<td>4.30E-01</td>
</tr>
<tr>
<td>1</td>
<td>-42.0426%</td>
<td>1,190.28</td>
<td>453</td>
<td>2.63E+00</td>
</tr>
<tr>
<td>2</td>
<td>-305.0355%</td>
<td>(2,096,943,689,911.92)</td>
<td>30,662,207,805,072</td>
<td>-6.84E-02</td>
</tr>
<tr>
<td>3</td>
<td>-298.1966%</td>
<td>(758,378,102,955.59)</td>
<td>11,471,817,147,157</td>
<td>-6.61E-02</td>
</tr>
<tr>
<td>4</td>
<td>-291.5858%</td>
<td>(274,274,032,389.87)</td>
<td>4,292,013,875,073</td>
<td>-6.39E-02</td>
</tr>
<tr>
<td>5</td>
<td>-285.1955%</td>
<td>(99,193,562,252.03)</td>
<td>1,605,795,098,231</td>
<td>-6.18E-02</td>
</tr>
<tr>
<td>6</td>
<td>-279.0183%</td>
<td>(35,874,195,371.97)</td>
<td>600,785,234,349</td>
<td>-5.97E-02</td>
</tr>
<tr>
<td>7</td>
<td>-273.0470%</td>
<td>(12,974,204,330.23)</td>
<td>224,775,250,484</td>
<td>-5.77E-02</td>
</tr>
<tr>
<td>8</td>
<td>-267.2750%</td>
<td>(4,692,228,676.37)</td>
<td>84,096,488,526</td>
<td>-5.58E-02</td>
</tr>
<tr>
<td>9</td>
<td>-261.6954%</td>
<td>(1,696,982,831.08)</td>
<td>31,463,527,011</td>
<td>-5.39E-02</td>
</tr>
<tr>
<td>10</td>
<td>-256.3019%</td>
<td>(613,727,370.38)</td>
<td>11,771,646,881</td>
<td>-5.21E-02</td>
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<tr>
<td>11</td>
<td>-251.0883%</td>
<td>(221,959,165.01)</td>
<td>4,404,205,991</td>
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</tr>
<tr>
<td>12</td>
<td>-246.0486%</td>
<td>(80,273,022.90)</td>
<td>1,647,780,736</td>
<td>-4.87E-02</td>
</tr>
<tr>
<td>13</td>
<td>-241.1770%</td>
<td>(29,031,096.04)</td>
<td>616,502,583</td>
<td>-4.71E-02</td>
</tr>
<tr>
<td>14</td>
<td>-236.4680%</td>
<td>(10,499,047.87)</td>
<td>230,664,295</td>
<td>-4.55E-02</td>
</tr>
<tr>
<td>15</td>
<td>-231.9163%</td>
<td>(3,796,787.67)</td>
<td>86,308,448</td>
<td>-4.40E-02</td>
</tr>
<tr>
<td>16</td>
<td>-227.5172%</td>
<td>(1,372,863.36)</td>
<td>32,299,962</td>
<td>-4.25E-02</td>
</tr>
<tr>
<td>17</td>
<td>-223.2669%</td>
<td>(496,232.55)</td>
<td>12,093,726</td>
<td>-4.10E-02</td>
</tr>
<tr>
<td>18</td>
<td>-219.1637%</td>
<td>(179,192.48)</td>
<td>4,534,159</td>
<td>-3.95E-02</td>
</tr>
<tr>
<td>19</td>
<td>-215.2116%</td>
<td>(64,533.12)</td>
<td>1,706,196</td>
<td>-3.78E-02</td>
</tr>
<tr>
<td>20</td>
<td>-211.4293%</td>
<td>(23,067.51)</td>
<td>648,544</td>
<td>-3.56E-02</td>
</tr>
</tbody>
</table>
And after 28 iterations we end up with a solution for IR= -200.33%, while the correct solution is 8.97%. What exactly happen here?

If we take a look at the f(i) between -15% to 12% we get the following plot of the function f(i):

With a starting guess of $i_0=1\%$ we see that the tangent $f'(i)$ is positive and we get a huge steps towards the left and our next iteration starting point is $i_1=-42\%$. Since $f'(i)$ at that point is also positive (see previous table) we get another huge step towards left at we end up with $i_2=-305\%$, now $f(i_2)$ turns negative while $f'(i_2)$ remains positive and we now move towards the right to $i_3=-298\%$. This continue until $i_{28}$ where we get $f(i_{28})=0$ and the iteration stop at $i_{28}=-200.33\%$. Clearly at starting guess of $i_0=1\%$ was not desirable since we didn’t get to the right solution for $i$.

Now let’s see what happen if we instead choose a starting guess of $i_0=3\%$. 
The Fundamental Financial Equation

<table>
<thead>
<tr>
<th>n</th>
<th>I</th>
<th>F(i)</th>
<th>F'(i)</th>
<th>dx</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.000%</td>
<td>2,621.50</td>
<td>(1,970)</td>
<td>-1.33E+00</td>
</tr>
<tr>
<td>1</td>
<td>136.0842%</td>
<td>(130,892,094,140,252.00)</td>
<td>(1,670,010,893,194,660)</td>
<td>7.84E-02</td>
</tr>
<tr>
<td>2</td>
<td>128.2464%</td>
<td>(47,333,112,076,215.30)</td>
<td>(624,880,411,164,816)</td>
<td>7.57E-02</td>
</tr>
<tr>
<td>3</td>
<td>120.6717%</td>
<td>(17,116,241,452,970.80)</td>
<td>(233,820,804,449,557)</td>
<td>7.32E-02</td>
</tr>
<tr>
<td>4</td>
<td>113.3514%</td>
<td>(6,189,302,659,306.10)</td>
<td>(87,494,317,620,514)</td>
<td>7.07E-02</td>
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<tr>
<td>5</td>
<td>106.2775%</td>
<td>(2,238,014,131,122.31)</td>
<td>(32,740,802,164,227)</td>
<td>6.84E-02</td>
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<td>6</td>
<td>99.4419%</td>
<td>(809,224,331,324.50)</td>
<td>(12,252,212,171,840)</td>
<td>6.60E-02</td>
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<tr>
<td>7</td>
<td>92.8372%</td>
<td>(292,587,963,047.92)</td>
<td>(4,585,210,252,243)</td>
<td>6.38E-02</td>
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<tr>
<td>8</td>
<td>86.4561%</td>
<td>(105,784,176,555.85)</td>
<td>(1,716,044,588,858)</td>
<td>6.16E-02</td>
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<td>9</td>
<td>80.2916%</td>
<td>(38,243,312,317.04)</td>
<td>(642,286,972,280)</td>
<td>5.95E-02</td>
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<tr>
<td>10</td>
<td>74.3374%</td>
<td>(13,824,597,902.70)</td>
<td>(240,419,607,864)</td>
<td>5.75E-02</td>
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<tr>
<td>11</td>
<td>68.5872%</td>
<td>(4,996,896,752.82)</td>
<td>(90,004,295,247)</td>
<td>5.55E-02</td>
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<tr>
<td>12</td>
<td>63.0354%</td>
<td>(1,805,856,746.36)</td>
<td>(33,699,733,930)</td>
<td>5.36E-02</td>
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<tr>
<td>13</td>
<td>57.6767%</td>
<td>(652,497,404.21)</td>
<td>(12,620,727,483)</td>
<td>5.17E-02</td>
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<tr>
<td>14</td>
<td>52.5067%</td>
<td>(235,697,010.71)</td>
<td>(4,727,956,311)</td>
<td>4.99E-02</td>
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<tr>
<td>15</td>
<td>47.5215%</td>
<td>(85,105,954.67)</td>
<td>(1,771,939,795)</td>
<td>4.80E-02</td>
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<td>16</td>
<td>42.7185%</td>
<td>(30,712,839.27)</td>
<td>(664,504,313)</td>
<td>4.62E-02</td>
</tr>
<tr>
<td>17</td>
<td>38.0966%</td>
<td>(11,074,152.25)</td>
<td>(249,437,881)</td>
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<tr>
<td>18</td>
<td>33.6569%</td>
<td>(3,987,693.84)</td>
<td>(93,775,190)</td>
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<tr>
<td>19</td>
<td>29.4045%</td>
<td>(1,432,775.58)</td>
<td>(35,344,404)</td>
<td>4.05E-02</td>
</tr>
<tr>
<td>20</td>
<td>25.3508%</td>
<td>(512,818.14)</td>
<td>(13,382,081)</td>
<td>3.83E-02</td>
</tr>
<tr>
<td>21</td>
<td>21.5187%</td>
<td>(182,228.85)</td>
<td>(5,110,634)</td>
<td>3.57E-02</td>
</tr>
<tr>
<td>22</td>
<td>17.9530%</td>
<td>(63,817.70)</td>
<td>(1,986,378)</td>
<td>3.21E-02</td>
</tr>
<tr>
<td>23</td>
<td>14.7402%</td>
<td>(21,646.29)</td>
<td>(802,037)</td>
<td>2.70E-02</td>
</tr>
<tr>
<td>24</td>
<td>12.0413%</td>
<td>(6,802.04)</td>
<td>(352,742)</td>
<td>1.93E-02</td>
</tr>
<tr>
<td>25</td>
<td>10.1130%</td>
<td>(1,751.38)</td>
<td>(186,348)</td>
<td>9.40E-03</td>
</tr>
<tr>
<td>26</td>
<td>9.1731%</td>
<td>(260.15)</td>
<td>(133,446)</td>
<td>1.95E-03</td>
</tr>
<tr>
<td>27</td>
<td>8.9782%</td>
<td>(9.09)</td>
<td>(124,213)</td>
<td>7.32E-05</td>
</tr>
<tr>
<td>28</td>
<td>8.9709%</td>
<td>(0.01)</td>
<td>(123,877)</td>
<td>9.93E-08</td>
</tr>
<tr>
<td>29</td>
<td>8.9708%</td>
<td>(0.00)</td>
<td>(123,876)</td>
<td>1.83E-13</td>
</tr>
</tbody>
</table>

This time we get to the right solution after 29 iteration at $i_{29}=8.97\%$, but not without wild swing of interest I, going from 3% to 136% at the first iteration and slowly going back to the correct solution after 29 iterations. This is also very classic behavior since we observed that $f(i)$ goes towards larger negative number and the tangent $(f'(i))$ goes close to parallel with $f(x)$ and therefore it only procedure small steps back towards the right solution.

I like this example since it highlight two potential issues with the newton iteration:

1) First a poorly choose starting point can set us of in the wrong direction resulting in a false solution to the problem.

2) Controlling the step size is needed to avoid wild swing and slow convergence to the right solution.
In order to cope with this we modify the newton iteration by applying the following safeguards:

1) We determine which direction the iteration are going by calculated NP\textsubscript{0} using an interest of 0. If NP\textsubscript{0}<NP) then the direction is towards positive interest. If on the other hand NP\textsubscript{0}>NP then the direction is towards negative interest.

2) We then chose i\textsubscript{0} using some approximately value and ensure if direction is less than 0 then the starting value should also be less than 0 otherwise positive interest as the starting point.

3) If a step size put us in the wrong direction we simply ignore the iterations step and multiply the original start guess with 2. If the next step also put us in the wrong direction we now multiply the original start guess with 4, then 8 etc. until we are moving in the right direction.

4) If a step size \( di_{n} = \frac{f(i_{n})}{f'(i_{n})} \) result in an increase of the function value \( f(i_{n+1}) \): \( f(i_{n+1}) > f(i_{n}) \) then we continue to reduce the step \( di_{n} = di_{n}/2 \) until \( f(i_{n+1}) \leq f(i_{n}) \) or we have done it 3 times in a row. In which case we stop and use the new i\textsubscript{n+1} as the new starting point for the next iteration.

This algorithm works pretty efficient and we get the correct solution after only 7 iterations as see below:

<table>
<thead>
<tr>
<th>n</th>
<th></th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.6800%</td>
<td>Safeguards rule 1. Doubling initial guess of 1.68%</td>
</tr>
<tr>
<td>1</td>
<td>3.3640%</td>
<td>Safeguards rule 2. Reduced step size with a factor 8</td>
</tr>
<tr>
<td>2</td>
<td>10.9550%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>9.5152%</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9.0211%</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>8.9713%</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>8.9708%</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>8.9708%</td>
<td></td>
</tr>
</tbody>
</table>

I have seen many financial calculators that sometimes goes wrong in the iterations process for finding the interest rate. With these safeguards mention above you will get a lot more robust and accurate solution. All the above algorithm has been implemented in the financial calculator found on [http://www.hvks.com/Numerical/webfinance.html](http://www.hvks.com/Numerical/webfinance.html)
Reference

1. GnuCash – Chapter 8.4 Calculation