The Fundamental Financial Equation.

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Abstract:

The Fundamental Financial Equation links the five variables. Present value (PV), Future value (FV), period Payment (PMT), number of periods (NP), and the interest rate (IR) together in this equation:

$$\left(PV+\frac{PMT(1+c\cdot IR)}{IR}\right)((1+IR)^{NP}-1)+PV+FV=0$$

That can be useful in several scenarios, including mortgage calculation. This paper highlights the equation and devises a method for calculating all the other variables when only four of the five variables are known, Including the Interest rate based on iteration.

Introduction to the Fundamental Financial Equation

Understanding how the value of money changes over time is essential in finance. This understanding underpins personal financial decisions, corporate finance, and investment strategies. The Fundamental Financial Equation provides a comprehensive formula that connects five crucial financial variables: Present Value (PV), Future Value (FV), Periodic Payment (PMT), Number of Periods (NP), and Interest Rate (IR).

This equation is the foundation of the time value of money and is a tool for solving various financial calculations. It allows determining one unknown variable when the other four are specified, making it useful across multiple financial disciplines.

Applications of the Fundamental Financial Equation

The versatility of the Fundamental Financial Equation allows it to be applied across a wide range of scenarios, including:

Regular Mortgage Payments Enables homebuyers to calculate their monthly mortgage payments by adjusting down payment, loan amount, loan term, and interest rate to fit their financial planning.
Balloon Payments are helpful when loans are structured with a large payment at the end of the financing period, commonly in commercial real estate.
Investment Planning. Investors utilize them to predict the future value of their current investments under various interest scenarios, aiding in strategic portfolio planning.

Retirement Planning helps individuals determine the periodic investments needed to accumulate a desired retirement fund, considering expected returns.
Corporate Finance is where companies apply the equation to assess the viability of long-term projects through present value calculations of anticipated cash flows, aiding capital budgeting decisions.

Setting the Stage for Future Chapters

This paper will further explore these applications, providing detailed examples to demonstrate the equation's practical application in different financial scenarios. We will discuss particular cases, such as when certain variables are zero, and their practical implications. Additionally, interest compounding and payment frequencies, crucial for accurate financial calculations, will be thoroughly examined. By the end of this paper, readers will gain a profound understanding of how to effectively utilize the Fundamental Financial Equation in their personal and professional financial use.

Contents

Introduction to the Fundamental Financial Equation

The Fundamental Financial equation is a financial equation that links the Present Value (PV) of a loan, investment, starting balance, etc., with the Future Value (FV), with a periodic payment (PMT), with the number of periods (NP) to an interest rate of (IR) through the equation:

$$\left(PV + \frac{PMT(1+c\cdot IR)}{IR}\right)((1+IR)^{NP}-1) + PV + FV = 0 \qquad (1)$$

The variables are:

- PV – Present Value of a starting balance of a bank account, Mortgage, Investment, etc.
- FV – Future Value of the above starting balance.
- NP – Number of periods. It can be Years, quarters, months, or daily.
- PMT – Periodic payment over several periods.
- IR – Interest rate per period.
- c – Is a constant that is 1 for interest added at the beginning of a period and 0 for the end.

The PV, FV, and PMT can be either 0 positive or negative. The sign is used to either indicate taking it out of the account (positive) or negative for paying into it. The result is the same regardless as long as it is done consistently. So, reversing the sign of PF, FV, and PMT yields the same result.

The above formula can also be written as:

$$PV(1+IR)^{NP} + PMT\ (1+c\cdot IR)((1+IR)^{NP}-1)/IR + FV = 0 \qquad (2a)$$

Or

$$\frac{PV*IR(1+IR)^{NP} + PMT\ (1+cIR)\left((1+IR)^{NP}-1\right)+FV*IR)}{IR} = 0 \qquad (2b)$$

And other variations. But the basic underlying equation is all the same.

Knowing four of the five variables, you can always calculate the 5th variable.

To make it easy, we create the following entity:

$$A = (1+IR)^{NP} - 1 \qquad (3)$$

And

$$B = \frac{1+c*IR}{IR} = \frac{1}{IR} + c \tag{4}$$

Substituting (3) and (4) above into (1), you get:

$$PV + (PMT * B) * A + PV + FV = 0 \tag{5}$$

Which solves each variable and gives the following equations:

$$PV = -\frac{FV+A*PMT*B}{A+1}\ Conditions\ IR \neq -1 \text{ (-100\%)} \tag{6}$$

$$FV = -(PV + A(PV + PMT * B)) \tag{7}$$

$$PMT = -\frac{FV+PV(A+1)}{A*B}\ Conditions\ IR \neq -1\ (-100\%)\ and\ c = 1 \tag{8}$$

$$NP = \frac{\log\left(\frac{PMT*B-FV}{PMT*B+PV}\right)}{\log(1+IR)} = \frac{\log(PMT*B-FV)-\log(PMT*B+PV)}{\log(1+IR)} \tag{9}$$

Unfortunately, we would need iteration to find the IR based on our fundamental financial equation. More about that later.

Looking at equation (1), the only troubling part is the division with IR when IR=0. We will look closer at that, particularly in case 1 below.

Looking at (6), we see that we divide with A+1 however $A + 1 = (1 + IR)^{NP} - 1 + 1 = (1 + IR)^{NP}$ Which is always ≠ 0 except for IR=-1. In (8), we have a similar problem since we divide by A·B, meaning neither A nor B can be zero. A is zero for IR=0 since $A = (1 + IR)^{NP} - 1 = (1 + 0)^{NP} - 1 = 0$. B can be zero only for IR=-1 and c=1. IR=0 is undefined, but it was already stated that (4) only works for IR≠0.

Lastly, in (9), we have that neither log(1+IR)≠0 nor $\log\left(\frac{PMT*B-FV}{PMT*B+PV}\right) \neq 0$. In the first case, we have that IR=0 yields log(1+0)=0, and for IR≤-1, the log is undefined since the argument to the log() is ≤0 where the log() function is undefined, at least in the real domain. In the second case, we need $\frac{PMT*B-FV}{PMT*B+PV} > 0$; otherwise, NP can't be calculated.

Special case 1 (IR=0 and PMT ≠ 0):

Now let's consider a particular case where IR=0 and PMT is not 0, and we quite easily find

$$PV + NP * PMT + FV = 0 \tag{10}$$

Which solves for PV, FV, PMT, and NP give:

$$PV = -(FV + NP * PMT) \tag{11}$$

$$FV = -(PV + NR * PMT) \tag{12}$$

$$PMT = -\frac{PV+FV}{NP} \tag{13}$$

$$NP = -\frac{PV+FV}{PMT} \tag{14}$$

Special case 2 (IR≠0 and PMT=0)

Another particular case is where IR is not equal to 0, and PMT is 0:

$$(PV(1 + IR)^{NP} - 1) + PV + FV = 0 \tag{15}$$

Solves for PV, FV, NP, and IR were given using the substitution at (3):

$$PV = -\frac{FV}{1+A} \tag{16}$$

$$FV = -PV(1 + A) \tag{17}$$

$$NP = \frac{\log\left(-\frac{FV}{PV}\right)}{\log(1+IR)} \tag{18}$$

$$IR = \left(\frac{FV}{PV}\right)^{\frac{1}{NP}} - 1;\ Condition\colon PV \cdot FV < 0) \tag{19}$$

Special case 3 (IR=0 and PMT=0)

This case is not relevant since you have PV+FV=0, and regardless of the number of periods, you will also have that PV=-FV or FV=-PV

Interest Compounding & Period Payment Frequency

The interest rate IR is usually specified in annual interest rate in %. However, as you have seen from your mortgage, these are not compounding yearly but on a monthly, quarterly, or even continuous compounding rate. We have also learned from the period payment PMT that, typically, your mortgage is done every month or quarter. Still, other payment rates can also be seen as quarterly, half-year, etc., so we will introduce two extra variables, Periodic Payment frequency (PF) and Interest Compounding frequency (CF), and then distinguish between continuous compounding and discrete compounding with the frequency of CF.

To convert from the nominal interest (IN) or annual interest rate to the IR in our formula (sometimes it is called the effective interest rate), we use the conversion below:

Continuous compounding:

$$IR = \exp\left(\frac{IN}{PF}\right) - 1 \tag{20}$$

Where IN is the nominal or annual rate of interest.

For discrete compounding, we need to take both the Compounding frequency CF and Payment frequency PF into account:

$$IR = \left(1 + \frac{IN}{CF}\right)^{\frac{CF}{PF}} - 1 \tag{21}$$

And to convert back to the nominal interest IN, we get:
Continuous compounding:

$$IN = \ln\left((1 + IR)^{PF}\right) \tag{22}$$

And for discrete compounding:

$$IN = CF\left((1 + IR)^{\frac{PF}{CF}} - 1\right) \tag{23}$$

Example for IR calculation

Let's see how it works out using an example. Assuming you use discrete compounding and a monthly frequency for both CF and PF and a nominal interest rate IN=6% Since there are 12 months in a year, you have:

$$PF = 12, CF = 12, IN = 6\% = 0.06$$

Using (21), you get:

$$IR = \left(1 + \frac{0.06}{12}\right)^{\frac{12}{12}} - 1 = 0.005$$

And convert it back using the values.

$$IR = 0.005, CF = 12, PF = 12$$

And discrete compounding (23) you get:

$$IN = 12\left((1 + 0.005)^{\frac{12}{12}} - 1\right) = 0.06 = 6\%$$

Another example of using continuous compounding is:

$$PF = 12, IN = 8\% = 0.08$$

You get using (20):

$$IR = \exp\left(\frac{0.08}{12}\right) - 1 = 0.0067$$

And to convert it back using (22), you get:

$$IN = \ln\left((1 + 0.0067)^{12}\right) = 0.08 = 8\%$$

Example of Mortgage calculation of the monthly payment

What is your monthly payment on a fixed 30-year mortgage of $200,000 at an annual nominal rate of 6%? Since it is compounding monthly and the interest rate is accrued monthly at the end, we have

$$PF = 12, CF = 12, IN = 6\%, c = 0, PV = 200{,}000, FV = 0, NP = 30 * 12 = 360$$

We have in the previous example already calculated the IR=0.005
Using the formula (8): $PMT = -\frac{FV+PV(A+1)}{A*B}$ Where A is given by (3) and B in (4), we get:

$$A = (1 + 0.005)^{360} - 1 = 5.0225$$

$$B = \frac{1+0*0.005}{0.005} = 200$$

$$PMT = -\frac{0+200{,}000(5.0225+1)}{5.0225*200} = -1199.10$$

The monthly payment PMT is $1,199.10.

We get it if we change it to 20 years mortgage instead of 30 years.

$$NP = 20 * 12 = 240$$

$$A = (1 + 0.005)^{240} - 1 = 2.3102$$

$$B = \frac{1+0*0.005}{0.005} = 200$$

$$PMT = -\frac{0+200{,}000(2.3102+1)}{2.3102*200} = -1432.86$$

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The new monthly payment is $1432.86

Example of Mortgage calculation with a balloon payment at the end

This is the same example as the last one. However, instead of paying a monthly payment of $1,432.86 over 20 years, you have negotiated a lower monthly rate of $1,300 and will instead have to pay a balloon payment at the end after 20 years. What is the value of the balloon payment at the end?

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$$PF = 12, CF = 12, IN = 6\%, c = 0, PV = 200{,}000, PMT = -1300, NP = 20 * 12 = 240$$

FV is given by (7) as $FV = -(PV + A(PV + PMT * B))$
A & B has previously been found to be A=2.3102 and B=200, so we have:

$$FV = -\left(200{,}000 + 2.3102(200{,}000 - 1300 * 200)\right) = -61{,}388$$

The minus sign indicates that you owe the mortgage company $61,388 after 20 years. (The correct result is $61,387.73, but we only used four digits in the A calculation).

Example of Mortgage calculation with number of periods

What is your monthly payment on a fixed 30-year mortgage of $200,000 at an annual nominal rate of 6%? Since it is compounding monthly and the interest rate is accrued monthly at the end, we have

$$PF = 12, CF = 12, IN = 6\%, c = 0, PV = 200{,}000, FV = 0, NP = 30 * 12 = 360$$

We have, in a previous example, calculated the monthly payment to be $1199.10

What happens if we instead increase our monthly payment to $1300?

In the previous example, we calculated the IR=0.005, and the new PMT is -1300.
Using the formula (9): $NP = \frac{\log\left(\frac{PMT*B-FV}{PMT*B+PV}\right)}{\log(1+IR)}$ And B is given in (4), we get:

$$B = \frac{1+0*0.005}{0.005} = 200$$

$$NP = \frac{\log\left(\frac{-1300*200-0}{-1300*200+200000}\right)}{\log(1+0.005)} = \frac{\log(4.3333)}{\log(1.005)} = 294$$

The new number of periods is now 294 months, or a shortening of the 30-year fixed mortgage with 5½ years.

Finding the nominal interest rate per year through iteration

This is the most interesting chapter and the reason I wrote this paper. In our fundamental equation:

$$\left(PV + \frac{PMT(1 + c \cdot IR)}{IR}\right)((1 + IR)^{NP} - 1) + PV + FV = 0$$

There is no way to find the interest IR as an equation, so we need to resort to the classic Newton iterations or some other method to solve the equation for IR. Newton Iterations are the number of repeated uses of the below iterations.

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \text{ Or } x_{n+1} = x_n - d_n \ where\ d_n = \frac{f(x)}{f'(x)}$$

Before we can use this, we need to rework our equation to make it easier to apply the Newton iteration. Substitute x for IR and name IR=i. We get:

$$f(i) = \left(PV + \frac{PMT(1 + c \cdot i)}{i}\right)((1 + i)^{NP} - 1) + PV + FV$$

Which can also be rewritten into an alternative form as:

$$f(i) = \frac{PMT(1 + c \cdot i)(1 + i)^{NP} - PMT(1 + c \cdot i) + i \cdot FV + i \cdot PV(1 + i)^{NP}}{i}$$

Now with f(i) above, we can find the derivate as:

$$f'(i) = \frac{((1+i)^{NP}-1)(-PMT(1+c\cdot i)) + NP\cdot i(1+i)^{(NP-1)}\ (PMT(1+c\cdot i)+i\cdot PV)}{i^2}$$

We therefore get:

$$i_{n+1} = i_n + \frac{\frac{PMT(1+c\cdot i)(1+i)^{NP} - PMT(1+c\cdot i) + i\cdot FV + i\cdot PV(1+i)^{NP}}{i_n}}{\frac{((1+i)^{NP}-1)(-PMT(1+c\cdot i)) + N\ \ \cdot i(1+i)^{(NP-1)}\ (PMT(1+c\cdot i)+i\cdot PV)}{i_n{}^2}}$$

Or

$$i_{n+1} = i_n + i_n \frac{PMT(1+c\cdot i)(1+i)^{NP} - PMT(1+c\cdot i) + i\cdot FV + i\cdot PV(1+i)^{NP}}{((1+i)^{NP}-1)(-PMT(1+c\cdot i)) + NP\cdot i(1+i)^{(NP-1)}\ (PMT(1+c\cdot i)+iPV)}$$

To see how it works in real life, we can try to find interest in the following scenario.

Example: A 30-year maturity bond with a face value of $1000. The coupon rate is 8% annually. It sells for $900. Interest is accrued at the end of the period. We therefore have

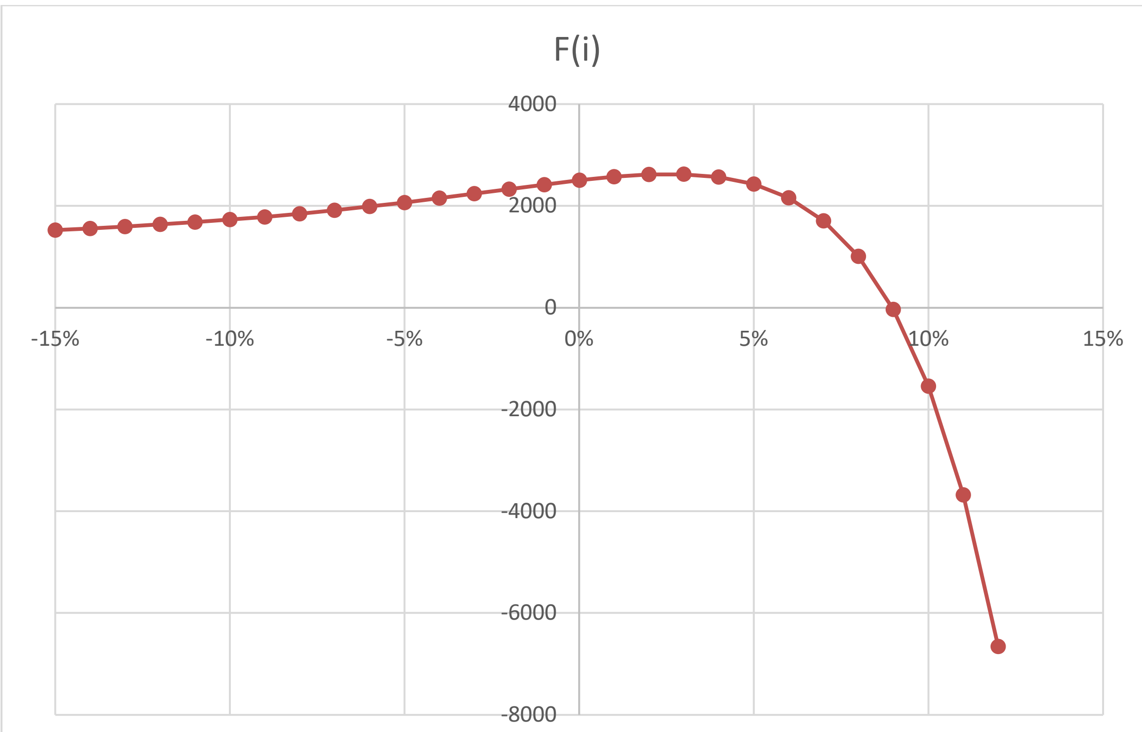
$$NP = 30, FV = 1000, PV = -900, PMT = 8\%\ of\ \$1000 = 80,$$

Start with a guess of i_0=1% and get the following iterations.

n	I	F(i)	F'(i)	dx
0	1.000%	2,569.73	5,970	4.30E-01
1	-42.0426%	1,190.28	453	2.63E+00
2	-305.0355%	(2,096,943,689,911.92)	30,662,207,805,072	-6.84E-02
3	-298.1966%	(758,378,102,955.59)	11,471,817,147,157	-6.61E-02
4	-291.5858%	(274,274,032,389.87)	4,292,013,875,073	-6.39E-02
5	-285.1955%	(99,193,562,252.03)	1,605,795,098,231	-6.18E-02
6	-279.0183%	(35,874,195,371.97)	600,785,234,349	-5.97E-02
7	-273.0470%	(12,974,204,330.23)	224,775,250,484	-5.77E-02
8	-267.2750%	(4,692,228,676.37)	84,096,488,526	-5.58E-02
9	-261.6954%	(1,696,982,831.08)	31,463,527,011	-5.39E-02
10	-256.3019%	(613,727,370.38)	11,771,646,881	-5.21E-02
11	-251.0883%	(221,959,165.01)	4,404,205,991	-5.04E-02
12	-246.0486%	(80,273,022.90)	1,647,780,736	-4.87E-02
13	-241.1770%	(29,031,096.04)	616,502,583	-4.71E-02
14	-236.4680%	(10,499,047.87)	230,664,295	-4.55E-02
15	-231.9163%	(3,796,787.67)	86,308,448	-4.40E-02
16	-227.5172%	(1,372,863.36)	32,299,962	-4.25E-02
17	-223.2669%	(496,232.55)	12,093,726	-4.10E-02
18	-219.1637%	(179,192.48)	4,534,159	-3.95E-02
19	-215.2116%	(64,533.12)	1,706,196	-3.78E-02
20	-211.4293%	(23,067.51)	648,544	-3.56E-02
21	-207.8725%	(8,076.40)	253,329	-3.19E-02

22	-204.6844%	(2,669.13)	106,212	-2.51E-02
23	-202.1714%	(750.25)	52,536	-1.43E-02
24	-200.7433%	(133.82)	34,945	-3.83E-03
25	-200.3604%	(7.11)	31,296	-2.27E-04
26	-200.3377%	(0.02)	31,092	-7.48E-07
27	-200.3376%	(0.00)	31,091	-8.08E-12
28	-200.3376%	(0.00)	31,091	-1.17E-16

After 28 iterations, we end up with a solution for IR= -200.33%, while the correct solution is 8.97%. What exactly happens here?
If we take a look at the f(i) between -15% to 12%, we get the following plot of the function f(i):



With a starting guess of i_0=1%, we see that the tangent of f' (i) is positive, and we get a huge step towards the left, and our next iteration starting point is i_1=-42%. Since f' (i) at that point is also positive (see previous table), we get another huge step towards left as we end up with i_2=-305%, now f(i_2) turns negative while f' (i_2) remains positive and we now move towards the right to i_3=-298%. This continues until i_{28} where we get f(i_{28})=0, and the iteration stops at i_{28}=-200.33%. A starting guess of i_0=1% was undesirable since we didn't find the right solution for i.

Let's see what happens if we choose a starting guess of i_0=3% instead.

n	**I**	**F(i)**	**F'(i)**	**dx**

0	3.000%	2,621.50	(1,970)	-1.33E+00
1	136.0842%	(130,892,094,140,252.00)	(1,670,010,893,194,660)	7.84E-02
2	128.2464%	(47,333,112,076,215.30)	(624,880,411,164,816)	7.57E-02
3	120.6717%	(17,116,241,452,970.80)	(233,820,804,449,557)	7.32E-02
4	113.3514%	(6,189,302,659,306.10)	(87,494,317,620,514)	7.07E-02
5	106.2775%	(2,238,014,131,122.31)	(32,740,802,164,227)	6.84E-02
6	99.4419%	(809,224,331,324.50)	(12,252,212,171,840)	6.60E-02
7	92.8372%	(292,587,963,047.92)	(4,585,210,252,243)	6.38E-02
8	86.4561%	(105,784,176,555.85)	(1,716,044,588,858)	6.16E-02
9	80.2916%	(38,243,312,317.04)	(642,286,972,280)	5.95E-02
10	74.3374%	(13,824,597,902.70)	(240,419,607,864)	5.75E-02
11	68.5872%	(4,996,896,752.82)	(90,004,295,247)	5.55E-02
12	63.0354%	(1,805,856,746.36)	(33,699,733,930)	5.36E-02
13	57.6767%	(652,497,404.21)	(12,620,727,483)	5.17E-02
14	52.5067%	(235,697,010.71)	(4,727,956,311)	4.99E-02
15	47.5215%	(85,105,954.67)	(1,771,939,795)	4.80E-02
16	42.7185%	(30,712,839.27)	(664,504,313)	4.62E-02
17	38.0966%	(11,074,152.25)	(249,437,881)	4.44E-02
18	33.6569%	(3,987,693.84)	(93,775,190)	4.25E-02
19	29.4045%	(1,432,775.58)	(35,344,404)	4.05E-02
20	25.3508%	(512,818.14)	(13,382,081)	3.83E-02
21	21.5187%	(182,228.85)	(5,110,634)	3.57E-02
22	17.9530%	(63,817.70)	(1,986,378)	3.21E-02
23	14.7402%	(21,646.29)	(802,037)	2.70E-02
24	12.0413%	(6,802.04)	(352,742)	1.93E-02
25	10.1130%	(1,751.38)	(186,348)	9.40E-03
26	9.1731%	(260.15)	(133,446)	1.95E-03
27	8.9782%	(9.09)	(124,213)	7.32E-05
28	8.9709%	(0.01)	(123,877)	9.93E-08
29	8.9708%	(0.00)	(123,876)	1.83E-13

This time, we get to the right solution after 29 iterations at i_{29}=8.97%, but not without a wild swing of interest I, going from 3% to 136% at the first iteration and slowly returning to the correct solution after 29 iterations. This is also very classic behavior since we observed that f(i) goes towards a larger negative number and the tangent (f' (i)) goes close to parallel with f(x), and therefore it only procedure small steps back towards the right solution.

I like this example since it highlights two potential issues with the Newton iteration:

1) First, a poorly chosen starting point can set us off in the wrong direction, resulting in a false solution to the problem.
2) Controlling the step size is needed to avoid wild swing and slow convergence to the right solution

To cope with this, we modify the Newton iteration by applying the following safeguards:

1) We determine which direction the iteration goes by calculating NP_0 using an interest of 0. If $NP_0<NP$), then the direction is towards positive interest. If, on the other hand, $NP_0>NP$, then the direction is toward negative interest.
2) We then chose i_0 using some approximate value and ensured that if the direction is less than 0, the starting value should also be less than 0; otherwise, positive interest is the starting point.
3) If a step size puts us in the wrong direction, we ignore the iterations step and multiply the original start guess with 2. If the next step also puts us in the wrong direction, we multiply the original start guess with 4, then 8, etc., until we move in the right direction.
4) If a step size $di_n = \frac{f(i_n)}{f'(i_n)}$ results in an increase of the function value $f(i_{n+1})$: $f(i_{n+1}) > f(i_n)$ then we continue to reduce the step $di_n = di_n/2$ until $f(i_{n+1}) \leq f(i_n)$, or we have done it 3 times in a row. In this case, we stop and use the new i_{n+1} as the new starting point for the next iteration.

This algorithm works pretty efficiently, and we get the correct solution after only 7 iterations, as seen below:

n	I	Comment
0	1.6800%	
1	3.3640%	Safeguards rule 1. Doubling the initial guess of 1.68%
2	10.9550%	Safeguards rule 2. Reduced step size with a factor of 8
3	9.5152%	
4	9.0211%	
5	8.9713%	
6	8.9708%	
7	8.9708%	

I have seen many financial calculators that sometimes go wrong in the iteration process when finding the interest rate. You will find a more robust and accurate solution with the abovementioned safeguards.

Description of the Bisection Method for Finding Interest Rate

The bisection method provides an alternative method to Newton's method. It is a robust and straightforward numerical approach for finding the interest rate (IR) when solving financial equations that cannot be rearranged algebraically to isolate IR. This is an alternative to the Newton method and is simpler to implement. This technique is especially valuable in contexts where the equation is non-linear, and other methods, like Newton's iteration, may fail to converge or require good initial guesses to ensure convergence. Another benefit is that this method doesn't require access to the function's derivative compared to Newton's method.

Overview of the Bisection Method

The bisection method is a bracketing method for finding the root of a function. It works by repeatedly narrowing down the interval within which the root lies until it converges sufficiently close to the actual root. This method guarantees convergence to a root, provided the function changes sign over the interval and is continuous. One benefit is that we don't need the first derivative of the f(IR).

Applying Bisection to the Financial Equation

We used the formula for NP because it provides an easier explanation of how to find the IR—se formula (9).

$$NP = \frac{\log(PMT*B-FV)-\text{lo}\quad(PMT*B+PV)}{\log{(1+IR)}}$$

And if IR=0, then we use the simplified equation:

$$NP = -\frac{PV + FV}{PMT}$$

The first step is to find an interval where we know we can find IR. We start by setting IR to zero and calculating the new NP_{new}. If NP_{new} is less than the current NP, we know the IR must be greater than 0, and IR equal to zero represents our lower bound interval for IR. We then increase IR (in steps of 1% at a time) until the NP_{new} exceeds NP. Now, we have found our upper bound for the value of IR.

We then start our bisection method by choosing the midpoint between the two endpoints. If the new midpoint is still greater than NP, then the new upper bound is set to the midpoint; otherwise, we set the lower bound to the midpoint and then repeat the iterations until the difference between the two endpoints is less than 1e-5. (That is sufficient to guarantee at least 4 decimal points)

The bisection method is particularly useful when dealing with financial equations where parameters such as payments (PMT), present value (PV), future value (FV), and number of periods (NP) are known. Still, the interest rate (IR) is unknown. It avoids the pitfalls of derivative calculation required in Newton's method, making it more stable, though potentially slower.

The bisection method provides a reliable and simple approach to finding interest rates in complex financial scenarios. Systematic narrowing of the search interval ensures convergence to an accurate solution suitable for financial models where precision is crucial. This method has been integrated into financial calculators to provide consistent and dependable results in real-world applications.

Example:
PV=100,000, FV=-10,000, PMT=-5000, NP=30. What is the IR Rate?

By iteration, we start with IR=0 and get an NP'=18. This is below the required NP of 30. So, 0% is the lower bound. Continue adding 1% until you find the upper bound at IR=4%, where NP'=38.9. We start the bisection method with the interest interval [3%,4%]. After 10 iterations, you get the answer of 3.156%

Solution: IR found by iteration
Iteration 0: IR = [0.03000, 0.04000]
Iteration 1: IR = IR = [0.03000, 0.03500], diff = 5.00e-3
Iteration 2: IR = IR = [0.03000, 0.03250], diff = 2.50e-3
Iteration 3: IR = IR = [0.03125, 0.03250], diff = 1.25e-3
Iteration 4: IR = IR = [0.03125, 0.03188], diff = 6.25e-4
Iteration 5: IR = IR = [0.03125, 0.03156], diff = 3.13e-4
Iteration 6: IR = IR = [0.03141, 0.03156], diff = 1.56e-4
Iteration 7: IR = IR = [0.03148, 0.03156], diff = 7.81e-5
Iteration 8: IR = IR = [0.03152, 0.03156], diff = 3.91e-5
Iteration 9: IR = IR = [0.03154, 0.03156], diff = 1.95e-5
Iteration 10: IR = IR = [0.03155, 0.03156], diff = 9.77e-6
IR approximation after max iterations: 0.03156
IR calculation was successful, with the result of 3.156%

The starting interval will always be 1% wide, and it will be reduced to half the interval of the previous iteration.

We can estimate the number of iterations needed by solving the equation:

$$0.5^n=10^{-3}$$

Solving for n we get $n\cdot\log10(0.5)=\log10(10^{-3}) \Rightarrow n=-3/\log10(0.5)\sim10$ iterations in line with the above result.

All the above algorithms have been implemented in the financial calculator found at http://www.hvks.com/Numerical/webfinance.html

Reference

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